

# Asympmtotics of enumerative invariants in $\mathbb{C}P^2$

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## 1 Introduction

The GW-invariants were constructed first by Ruan-Tian for semi-positive symplectic manifolds (see [RT94], [RT97], [MS94]) and subsequently for general symplectic manifolds (see [LT96], [FO99] etc.). For any compact symplectic manifold  $M$  of dimension  $2n$ , its GW-invariants are given by a family of the following multi-linear maps:

$$\Phi_{g,a,k}^M : H^*(\mathcal{M}_{g,k}, \mathbb{Q}) \times H^*(M, \mathbb{Q})^k \mapsto \mathbb{Q}, \quad (1.1)$$

where  $a \in H_2(M, \mathbb{Z})$  and  $\mathcal{M}_{g,k}$  denotes the moduli of stable curves of genus  $g$  and with  $k$  marked points. For any  $\beta \in H^*(\mathcal{M}_{g,k}, \mathbb{Q})$  and  $\alpha_1, \dots, \alpha_k \in H^*(M, \mathbb{Q})$ , we have

$$\Phi_{g,a,k}^M(\beta; \alpha_1, \dots, \alpha_k) = \int_{\mathcal{M}^{vir}(g,a,k)} \mathbf{ev}^*(\beta \wedge \alpha_1 \wedge \dots \wedge \alpha_k),$$

where  $\mathcal{M}^{vir}(g, a, k)$  is the virtual moduli of stable maps of genus  $g$ , homology class  $a$  and  $k$  marked points and  $\mathbf{ev}$  is the evaluation map from stable maps to  $\mathcal{M}_{g,k} \times M^k$ . As usual, we consider the generating function

$$\mathbf{F}_g^M(w) = \sum_{a,k} \frac{\Phi_{g,a,k}^M(1; w, \dots, w)}{k!}, \quad w \in H^*(M, \mathbb{C}). \quad (1.2)$$

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For simplicity, we restrict  $\mathbf{F}_g^M$  to the even part  $H_{ev}(M)$  of  $H^*(M, \mathbb{C})$ . Let  $\gamma_0, \dots, \gamma_L$  be an integral basis for  $H_{ev}(M)$  such that each  $\gamma_i$  is of pure degree  $2d_i$  and  $0 = d_0 < 1 = d_1 \leq \dots \leq d_L = n$ . Write  $w = \sum t_i \gamma_i$ , then the restriction of  $\mathbf{F}_g$  is of the form

$$\mathbf{F}_g^M(t_0, \dots, t_L) = \sum_{a, k} \sum_{k_0 + \dots + k_L = k} \psi_{g,a}(k_0, \dots, k_L) t_1^{k_1} \dots t_L^{k_L},$$

where  $\psi_{g,a}(k_0, \dots, k_L)$  denotes the evaluation of  $\Phi_{g,a,k}^M$  at  $k_0$  of  $\gamma_0, \dots, \gamma_L$  divided by  $k_0! \dots k_L!$ . We observe that  $\psi_{g,a}(k_0, \dots, k_L) = 0$  unless

$$c_1(M)(a) + (3-n)(g-1) = \sum_{i=0}^L k_i(d_i - 1). \quad (1.3)$$

We further observe that  $\psi_{g,a}(k_0, \dots, k_L) = 0$  whenever  $k_0 > 0$  and  $a \neq 0$  and

$$\psi_{g,a}(0, k_1, \dots, k_{m-1}, k_m, \dots, k_L) = \left( \prod_{i=1}^{m-1} (\gamma_i(a))^{k_i} \right) \psi_{g,a}(0, \dots, 0, k_m, \dots, k_L),$$

where  $m \geq 2$  is chosen such that  $d_i = 1$  for  $1 \leq i \leq m-1$  and  $d_i \geq 2$  for any  $i \geq m$ .

The problem we concern is on the asymptotics of  $\psi_{g,a}$  as  $c_1(M)(a) \mapsto +\infty$ . More precisely, let  $k_m(\ell), \dots, k_L(\ell)$  satisfy: (1.3) holds with  $a$  replaced by  $\ell a$  and for any  $N > 0$  and  $i \geq m$ ,  $k_i(\ell) = \ell \sum_{j=0}^N c_{ij} \ell^{-j} + o(\ell^{-N})$  as  $\ell \rightarrow \infty$ . We expect that there are  $d, c$  and  $b_0, b_1, \dots$  such that

$$\psi_{g,\ell a}(0, 0, \dots, 0, k_m(\ell), \dots, k_L(\ell)) = \ell^d e^{-c\ell} \sum_{i=0}^N (b_i \ell^{-i} + o(\ell^{-N})). \quad (1.4)$$

A deeper problem is what geometric information is encoded in those coefficients  $b_0, b_1, \dots$ , in other words, can one express those  $b_i$  in terms of geometric quantities of  $M$ .

In this short paper, we study the first problem above on asymptotics in the case of  $M = \mathbb{CP}^2$  and  $g = 0, 1$ . Since  $H^0(\mathbb{CP}^2, \mathbb{Z}) = H^2(\mathbb{CP}^2, \mathbb{Z}) = H^4(\mathbb{CP}^2, \mathbb{Z}) = \mathbb{Z}$ , we can denote  $a$  by  $d \in \mathbb{Z}$  and have only one cohomology class  $\gamma_2$  of degree greater than 2. Thus it suffices to consider  $\psi_{g,d}(0, 0, 3d-1)$  in the above problem. For simplicity, we denote it by  $n_{g,d}$ . Note that

$$(3d-1)! n_{0,d} = \psi_{0,d}(0, 0, 3d-1).$$

It counts the number of rational curves of degree  $d$  in  $\mathbb{CP}^2$  through  $3d-1$  points in general position.

Our first main result is

**Theorem 1.1.** *In the case of genus  $g = 0$ , we have the following asymptotic expansion: There exists  $x_0, a_k^0 \in \mathbb{R}$  such that for any  $N \geq 4$ ,*

$$n_{0,d} = e^{-dx_0} \left( \sum_{k=3}^{N-1} a_k^0 d^{-k-1/2} + O(d^{-N-1/2}) \right).$$

**Corollary 1.1.** *Let  $n_{0,d}$  be the enumerative invariants as above for  $\mathbb{C}P^2$ , then*

$$\lim_{d \rightarrow \infty} \sqrt[d]{n_{0,d}} = e^{-x_0}. \quad (1.5)$$

In [FI94], (1.5) was claimed by P. Di Francesco and C. Itzykson without a valid proof. In [Zi11], A. Zinger verified some claims in [FI94] and proposed a proof of (1.5) under a conjectured condition which is still open.

Our second result is

**Theorem 1.2.** *In the case of genus  $g = 1$ , we have the following asymptotic expansion: There exists  $a_k^1 \in \mathbb{R}$  and  $x_0$  same as in Theorem 1.1 such that for any  $N \geq 4$ ,*

$$n_{1,d} = e^{-dx_0} \left( \frac{1}{48d} + \sum_{k=0}^{N-1} a_k^1 d^{-k-3/2} + O(d^{-N-3/2}) \right).$$

**Corollary 1.2.** *In case of  $\mathbb{C}P^2$ , we have*

$$\lim_{d \rightarrow \infty} \sqrt[d]{n_{1,d}} = \lim_{d \rightarrow \infty} \sqrt[d]{n_{0,d}} = e^{-x_0}. \quad (1.6)$$

This affirms a conjecture in [Zi11]. It is plausible that  $\sqrt[d]{n_{g,d}}$  converge to a fixed number independent of  $g$ . In general, we expect the same, i.e., the  $\ell$ -th root of  $\psi_{g,\ell a}(0, 0, \dots, 0, k_m(\ell), \dots, k_L(\ell))$  converges to a fixed number which is related to the convergent radius of the generating functions.

It remains to understand geometric information encoded in the coefficients  $a_k^0$  and  $a_k^1$  in the above expansions.

## 2 Consequences of the WDVV equation

In this section, we show some consequences of the WDVV equation. First we recall for any symplectic manifold  $M$ ,  $\mathbf{F}_0^M$  satisfies the WDVV equation:

$$\sum_{b,c} \frac{\partial^3 \mathbf{F}_0^M}{\partial t_i \partial j \partial b} \eta^{bc} \frac{\partial^3 \mathbf{F}_0^M}{\partial t_k \partial l \partial c} = \sum_{b,c} \frac{\partial^3 \mathbf{F}_0^M}{\partial t_i \partial l \partial b} \eta^{bc} \frac{\partial^3 \mathbf{F}_0^M}{\partial t_k \partial j \partial c}, \quad (2.1)$$

where  $i, j, k, l$  run over  $0, \dots, L$ .

If we let  $m \in [2, L]$  such that  $d_1 = 1$  for  $i < m$  and  $d_i > 1$  for  $i \geq m$ , then

$$\mathbf{F}_0^M(t_0, \dots, t_L) = p + \sum_{a \neq 0} \psi_{0,a}(0, \dots, 0, k_m, \dots, k_L) e^{\sum_{i=1}^{m-1} t_i \gamma_i(a)}, \quad (2.2)$$

where  $p$  is the homogeneous cubic polynomial in  $t_0, \dots, t_L$  which gives rise to the cup product on  $H_{ev}^*(M)$ . Also  $F_0$  satisfies the WDVV equation:

Now we assume  $M = \mathbb{C}P^2$ , then  $H_2(M, \mathbb{Z}) = \mathbb{Z}$  and (2.2) becomes

$$\mathbf{F}_0^{\mathbb{C}P^2}(t) = \frac{1}{2}(t_2 t_0^2 + t_0 t_1^2) + \sum_{k=1}^{\infty} n_{0,k} t_2^{3k-1} e^{kt_1}, \quad (2.3)$$

where  $t = (t_0, t_1, t_2)$ . We can further write

$$\mathbf{F}_0^{\mathbb{C}P^2} = \frac{1}{2}(t_2 t_0^2 + t_0 t_1^2) + t_2^{-1} F_0(t_1 + 3 \ln t_2), \quad (2.4)$$

where

$$F_0(z) = \sum_{d=1}^{\infty} n_{0,d} e^{dz}.$$

Using the WDVV equation which  $\mathbf{F} = \mathbf{F}_0^{\mathbb{C}P^2}$ , we have

$$\mathbf{F}_{112}^2 = \mathbf{F}_{111}\mathbf{F}_{122} + \mathbf{F}_{222}, \quad \mathbf{F}_{ijk} = \frac{\partial^3 \mathbf{F}}{\partial t_i \partial t_j \partial t_k}.$$

Direct computations show:

$$\mathbf{F}_{111} = t_2^{-1} F_0'''(t_1 + 3 \ln t_2), \quad \mathbf{F}_{112} = t_2^{-1} (3F_0''' - F_0'')(t_1 + 3 \ln t_2),$$

$$\mathbf{F}_{122} = t_2^{-1} (9F_0''' - 9F_0'' + 2F_0')(t_1 + 3 \ln t_2)$$

and

$$\mathbf{F}_{222} = t_2^{-1} (27F_0''' - 54F_0'' + 33F_0' - 6F_0)(t_1 + 3 \ln t_2).$$

It follows

$$(3F_0''' - F_0'')^2 = F_0'''(9F_0''' - 9F_0'' + 2F_0') + 27F_0''' - 54F_0'' + 33F_0' - 6F_0.$$

This can be simplified as

$$(27 + 2F_0' - 3F_0'')F_0''' = 6F_0 - 33F_0' + 54F_0'' + (F_0'')^2, \quad (2.5)$$

The following lemma is taken from [Zi11].

**Lemma 2.1.** *We have the estimates for  $n_{0,d}$ :*

$$\left(\frac{1}{27}\right)^d d^{-\frac{7}{2}} \leq n_{0,d} \leq 3 \left(\frac{4}{15}\right)^d d^{-\frac{7}{2}}. \quad (2.6)$$

*Proof.* We outline its proof by following [Zi11]. First we observe: If  $n_d$  is a sequence of numbers satisfying:

$$n_d = \sum_{d_1, d_2 \geq 0, d_1 + d_2 = d} \frac{f(d_1)f(d_2)}{f(d)} n_{d_1} n_{d_2}$$

for some  $f : \mathbb{Z} \mapsto \mathbb{R}^+$ , then

$$n_d = \frac{(2d-2)!}{d!(d-1)!} (f(1) n_1)^d, \quad d \geq 1.$$

It is proved in [RT94] by using the WDVV equation

$$n_{0,d} = \sum_{d_1, d_2 \geq 0, d_1 + d_2 = d} T(d_1, d_2) n_{0,d_1} n_{0,d_2},$$

where

$$T(d_1, d_2) = \frac{d_1 d_2 (3d_1 d_2 (d+2) - 2d^2)}{2(3d-3)(3d-2)(3d-1)}.$$

It is easy to see

$$\frac{f_1(d_1)f_1(d_2)}{f_1(d)} \leq T(d_1, d_2) \frac{f_2(d_1)f_2(d_2)}{f_2(d)},$$

where  $f_1(d) = d(3d-2)/54$  and  $f_2(d) = 2d^2/15$ . If we denote by  $n_d^j$  the sequence of numbers determined by the recursive formula for above  $n_d$ 's together with  $f$  replaced  $f_j$  and  $n_1^j = 1$ , then by induction, we can show  $n_1^1 \leq n_{0,d} \leq n_d^2$ . Therefore, we have

$$\frac{(2d-2)!}{d!(d-1)!} \left(\frac{1}{54}\right)^d \leq n_{0,d} \leq \frac{(2d-2)!}{d!(d-1)!} \left(\frac{2}{15}\right)^d.$$

The lemma follows from this and Stirling formula for  $d!$ .  $\square$

### 3 Proof of Theorem 1.1

First we observe that as a consequence of Lemma 2.1, we get a convergent radius  $x_0$  such that the power series  $\sum_{d=1}^{\infty} n_{0,d} e^{dz}$  converges for any  $z \in \mathbb{C}$  with  $\operatorname{Re}(z) < x_0$ , so  $F_0(z)$  is well-defined in the region  $\{z \mid \operatorname{Re}(z) < x_0\}$ .

Since  $n_{0,d} \geq 1$ , as observed in [Zi11], we can deduce

$$0 < F_0(z) < F_0'(z) < F_0''(z) < F_0'''(z), \quad \forall z \in (-\infty, x_0) \quad (3.1)$$

and

$$27 + 2F_0' - 3F_0'' > 0, \quad \forall z \in (-\infty, x_0). \quad (3.2)$$

By (3.1), the series for  $F_0$ ,  $F_0'$  and  $F_0''$  increase along  $(-\infty, x_0)$ . By (3.2),  $F_0 < F_0' < F_0'' < 27$  on  $(-\infty, x_0)$ . So  $F_0$ ,  $F_0'$  and  $F_0''$  converge at  $z = x_0$ . Moreover,  $27 + 2F_0' - 3F_0'' = 0$  at  $z = x_0$ , otherwise, (2.5) could be used to compute all derivatives of  $F_0$  at  $z = x_0$  and get a contradiction to the fact that  $x - 0$  is the convergent radius.

Clearly,  $F_0$  is analytic in  $\{\operatorname{Re} z < x_0\}$ ,

**Lemma 3.1.**  *$F_0$  can be analytically continued to  $\{\operatorname{Re} z < x_0 + \delta_0, 0 \leq \operatorname{Im} z \leq 2\pi\}$ , for some  $0 < \delta_0 < 1$ , i.e.,  $F_0$  is analytic in  $\{\operatorname{Re} z < x_0 + \delta_0, 0 < \operatorname{Im} z < 2\pi\}$ , and continuous in  $\{\operatorname{Re} z < x_0 + \delta_0, 0 \leq \operatorname{Im} z \leq 2\pi\}$ , and  $F_0$  has the expansion at  $x_0$ ,*

$$F_0(x_0 + re^{i\theta}) = \sum_{d=0}^{\infty} a_d r^{d/2} e^{i\theta d/2}, \quad \forall r < \delta_0, 0 \leq \theta \leq \pi, \quad (3.3)$$

$$F_0(x_0 + 2\pi i + re^{i\theta}) = \sum_{d=0}^{\infty} a_d r^{d/2} e^{i\theta d/2}, \quad \forall r < \delta_0, \pi \leq \theta \leq 2\pi, \quad (3.4)$$

where  $a_1 = a_3 = 0$ . Also we have  $|a_d| \delta_0^{d/2} \leq C$ .

*Proof.* Introduce new variables  $t, x, y, w$  such that

$$\frac{dt}{dz} = \frac{1}{27 + 2F'_0 - 3F''_0}, \quad x = 9F''_0 - 9F'_0 + 2F_0, \quad y = 3F''_0 - F'_0, \quad w = F''_0,$$

for example, we can take

$$\begin{aligned} t(z) &= \frac{z}{27} - \int_0^{+\infty} \left( \frac{1}{27 + 2F'_0(z-s) - 3F''_0(z-s)} - \frac{1}{27} \right) ds \\ x(z) &= \sum_{d=1}^{\infty} (3d-1)(3d-2) n_{0,d} e^{dz} \\ y(z) &= \sum_{d=1}^{\infty} d(3d-1) n_{0,d} e^{dz} \\ w(z) &= \sum_{d=1}^{\infty} d^2 n_{0,d} e^{dz}. \end{aligned}$$

In fact,

$$(\mathbf{F}_{22}, \mathbf{F}_{12}, \mathbf{F}_{11}) = (0, 0, t_0) + t_2^{-1} (x, y, w)(t_1 + 3 \ln t_2).$$

Then  $t \in \mathbb{R}$ ,  $x, y, w > 0$ , and  $x, y, w, t$  are strictly increasing on  $(-\infty, x_0)$ . Clearly,  $x, y, w \rightarrow 0$ ,  $t \rightarrow -\infty$  for  $z \rightarrow -\infty$ , in particular,  $t(z)$  has an inverse  $z(t)$  which maps  $(-\infty, t'_0)$  onto  $(-\infty, x_0)$ , where  $t'_0 = \lim_{z \rightarrow x_0} t(z)$ . Note that  $t'_0 < \infty$  since  $w$  blows up at finite time by the last equation in (3.5). Writing (2.5) in terms of the variable  $t$ , we get

$$\frac{d}{dt} \begin{pmatrix} x \\ y \\ w \\ z \end{pmatrix} = \begin{pmatrix} 27x + 4y^2 \\ 9x + 18y + 2yw \\ 3x + 6y + 9w + w^2 \\ 27 - 2y + 3w \end{pmatrix}. \quad (3.5)$$

Then (3.5) has a real analytic solution  $(\hat{x}, \hat{y}, \hat{w}, \hat{z})(t)$  in the maximal interval  $(-\infty, t_0)$  for some  $t_0 \in \mathbb{R}$  which extends the solution  $(x, y, w, z)$  on  $(-\infty, t'_0)$ , that is,

$$(x(z), y(z), w(z), z) = (\hat{x}, \hat{y}, \hat{w}, \hat{z})(t(z)), \quad \forall z \in (-\infty, x_0).$$

Using (3.5), we deduce that  $\hat{z} \in \mathbb{R}$ ,  $\hat{x}, \hat{y}, \hat{w}$  are strictly increasing positive functions on  $(-\infty, t_0)$ , and

$$\lim_{t \rightarrow t_0^-} \hat{x}(t) + \hat{y}(t) + \hat{w}(t) = +\infty.$$

Moreover, it follows from the equations on  $\hat{x}, \hat{y}, \hat{w}$  in (3.5)

$$\frac{d(\hat{x} + \hat{y} + \hat{w})}{dt} \leq (39 + 4\hat{y} + \hat{w})(\hat{x} + \hat{y} + \hat{w}).$$

It implies

$$\frac{d \log(\hat{x} + \hat{y} + \hat{w})}{dt} \leq 4(10 + \hat{y} + \hat{w}).$$

Hence, we have

$$\int_{t_0-1}^{t_0} (\widehat{y}(t) + \widehat{w}(t)) dt = +\infty. \quad (3.6)$$

Observe that

$$2y - 3w = 3F_0'' - 2F_0' = \sum_{d=1}^{\infty} d(3d-2) n_{0,d} e^{dz} > 0$$

and  $2y - 3w$  is strictly increasing on  $(-\infty, x_0)$ . It follows that  $(2\widehat{y} - 3\widehat{w})(t) > 0$  for  $t = t(z)$ ,  $z \in (-\infty, x_0)$  and  $(2\widehat{y} - 3\widehat{w})(t) \rightarrow 0$  for  $t \rightarrow -\infty$ . Using (3.5), we have

$$(2\widehat{y} - 3\widehat{w})' = 9\widehat{x} + (9 + \widehat{w})(2\widehat{y} - 3\widehat{w}) + 2\widehat{w}\widehat{y}. \quad (3.7)$$

Therefore,  $(2\widehat{y} - 3\widehat{w})(t) > 0$  and  $(2\widehat{y} - 3\widehat{w})'(t) > 0$  for  $t \in (-\infty, t_0)$ . Further, we derive from (3.7) the following differential inequality on  $(t_0 - 1, t_0)$ ,

$$(2\widehat{y} - 3\widehat{w})'(t) \geq (2\widehat{y} - 3\widehat{w})(t_0 - 1)\widehat{w}(t) + 2\widehat{w}(t_0 - 1)\widehat{y}(t).$$

Combining this with (3.6), we get

$$\lim_{t \rightarrow t_0^-} (2\widehat{y} - 3\widehat{w})(t) = +\infty,$$

so there exists a unique  $t_1 \in (-\infty, t_0)$  such that  $(2\widehat{y} - 3\widehat{w})(t_1) = 27$ . By (3.5), this implies that  $\widehat{z}'(t_1) = 0$  and  $\widehat{z}''(t_1) < 0$ . Since  $27 + 2F_0' - 3F_0'' = 0$  at  $z = x_0$ , we have  $\widehat{z}'(t(x_0)) = (27 + 2F_0' - 3F_0'')(x_0) = 0$ , so  $t_1 = t(x_0) = t_0'$ ,  $x_0 = \widehat{z}(t_1)$  and there is a  $\delta_1 > 0$  such that for  $|t| < \delta_1$ ,

$$\widehat{z}(t_1 + t) = x_0 + \sum_{k=2}^{\infty} b_k t^k, \quad \widehat{w}(t_1 + t) = \sum_{k=0}^{\infty} c_k t^k,$$

where  $b_2 < 0$ ,  $b_k, c_k \in \mathbb{R}$ . As  $\widehat{w}(t) = w(\widehat{z}(t))$ , there is a  $\delta_2 > 0$  such that for  $0 < z < \delta_2$ ,

$$w(x_0 - z) = \sum_{k=0}^{\infty} c'_k z^{\frac{k}{2}},$$

where  $c'_k \in \mathbb{R}$ . As  $w(z) = F_0''(z)$ , we have

$$F_0(x_0 - z) = F_0(x_0) - F_0'(x_0)z + \sum_{k=0}^{\infty} \frac{4c'_k}{(k+2)(k+4)} z^{\frac{k+4}{2}}.$$

Therefore (3.3) is true for  $\theta = \pi$  with

$$a_0 = F_0(x_0), \quad a_2 = F_0'(x_0), \quad a_1 = a_3 = 0, \quad a_k = \frac{4i^{-k}c'_{k-4}}{k(k-2)}, \quad \forall k \geq 4.$$

Since  $F_0$  is  $2\pi i$  periodic in  $\{\operatorname{Re} z < x_0\}$ , (3.4) is also true for  $\theta = \pi$ . Thus, by analytic continuation, (3.3), (3.4) hold for all  $\theta$  in the given range.  $\square$

**Remark 3.1.** *The analytic continuation of  $F_0$  in (3.3) and (3.4) is a more precise form of the following expansion:*

$$F_0(x_0 + z) = \sum_{d=0}^{\infty} a_d z^{d/2}, \quad \forall |z| < \delta_0, \quad a_1 = a_3 = 0.$$

*This was claimed in [FI94] without a proof. A justification was provided in [Zi11]. Our proof above is different and clearer.*

Now we complete the proof of Theorem 1.1. Fix any  $\delta \in (0, \delta_0)$ , using contour integration, we have

$$\begin{aligned} n_{0,d} &= \frac{1}{2\pi i} \int_{x_0}^{x_0+2\pi i} F_0(z) e^{-dz} dz \\ &= \frac{1}{2\pi i} \int_0^\delta F_0(x_0 + t) e^{-d(x_0+t)} dt \\ &\quad + \frac{1}{2\pi} \int_0^{2\pi} F_0(x_0 + \delta + it) e^{-d(x_0+\delta+it)} dt \\ &\quad - \frac{1}{2\pi i} \int_0^\delta F_0(x_0 + 2\pi i + t) e^{-d(x_0+2\pi i+t)} dt. \end{aligned} \quad (3.8)$$

One can easily have the following estimate:

$$\left| \frac{1}{2\pi} \int_0^{2\pi} F_0(x_0 + \delta + it) e^{-d(x_0+\delta+it)} dt \right| \leq C_1 e^{-d(x_0+\delta)}, \quad (3.9)$$

where  $C_1 = \max_{0 \leq t \leq \pi} |F_0(x_0 + \delta + it)| < +\infty$ . It follows from (3.3), (3.4) and the fact that  $a_1 = a_3 = 0$

$$\begin{aligned} &\frac{1}{2\pi i} \int_0^\delta F_0(x_0 + t) e^{-d(x_0+t)} dt - \frac{1}{2\pi i} \int_0^\delta F_0(x_0 + 2\pi i + t) e^{-d(x_0+2\pi i+t)} dt \\ &= \frac{1}{2\pi i} \int_0^\delta (F_0(x_0 + t) - F_0(x_0 + 2\pi i + t)) e^{-d(x_0+t)} dt \\ &= \frac{1}{2\pi i} \int_0^\delta \left( \sum_{k=0}^{\infty} a_k t^{k/2} - \sum_{k=0}^{\infty} a_k t^{k/2} (-1)^k \right) e^{-d(x_0+t)} dt \\ &= \sum_{k=3}^{\infty} \frac{a_{2k-1}}{\pi i} \int_0^\delta t^{k-1/2} e^{-d(x_0+t)} dt = e^{-dx_0} \sum_{k=3}^{\infty} \frac{a_{2k-1}}{\pi i} A(k, \delta, d), \end{aligned} \quad (3.10)$$

where

$$A(k, \delta, d) = \int_0^\delta t^{k-1/2} e^{-dt} dt = d^{-k-1/2} \int_0^{d\delta} t^{k-1/2} e^{-t} dt.$$

Clearly, we have

$$0 < A(k, \delta, d) < d^{-k-1/2} \int_0^{+\infty} t^{k-1/2} e^{-t} dt = \frac{\Gamma(k+1/2)}{d^{k+1/2}}$$



and

$$0 < A(k+1, \delta, d) < \delta A(k, \delta, d).$$

Fix any  $N \in \mathbb{Z}$  and  $n > 3$ , we have for  $3 \leq k < N$ ,

$$\begin{aligned} 0 < \Gamma(k+1/2) - d^{k+1/2} A(k, \delta, d) &= \int_{d\delta}^{+\infty} t^{k-1/2} e^{-t} dt \\ &\leq (d\delta)^{k-N} \int_{d\delta}^{+\infty} t^{N-1/2} e^{-t} dt < (d\delta)^{k-N} \Gamma(N+1/2), \end{aligned}$$

We further estimate

$$\begin{aligned} &\left| \sum_{k=3}^{\infty} \frac{a_{2k-1}}{\pi i} A(k, \delta, d) - \sum_{k=3}^{N-1} \frac{a_{2k-1} \Gamma(k+1/2)}{\pi i d^{k+1/2}} \right| \\ &\leq \sum_{k=N}^{\infty} \frac{|a_{2k-1}|}{\pi} A(k, \delta, d) + \sum_{k=3}^{N-1} \frac{|a_{2k-1}| (\Gamma(k+1/2) - d^{k+1/2} A(k, \delta, d))}{\pi d^{k+1/2}} \\ &\leq \sum_{k=N}^{\infty} \frac{|a_{2k-1}|}{\pi} \delta^{k-N} A(N, \delta, d) + \sum_{k=3}^{N-1} \frac{|a_{2k-1}| (d\delta)^{k-N} \Gamma(N+1/2)}{\pi d^{k+1/2}} \\ &\leq \sum_{k=N}^{\infty} \frac{|a_{2k-1}| \Gamma(N+1/2)}{\pi d^{N+1/2}} \delta^{k-N} + \sum_{k=3}^{N-1} \frac{|a_{2k-1}| \delta^{k-N} \Gamma(N+1/2)}{\pi d^{N+1/2}} \\ &= \frac{\Gamma(N+1/2)}{\pi d^{N+1/2}} \sum_{k=3}^{\infty} |a_{2k-1}| \delta^{k-N} = C_2 \frac{\Gamma(N+1/2)}{\pi (d\delta)^{N+1/2}}, \end{aligned} \tag{3.11}$$

where  $C_2 = \sum_{k=3}^{\infty} |a_{2k-1}| \delta^{k+1/2} < +\infty$ . It follows from (3.8), (3.9), (3.10) and (3.11)

$$\left| e^{dx_0} n_{0,d} - \sum_{k=3}^{N-1} \frac{a_{2k-1} \Gamma(k+1/2)}{\pi i d^{k+1/2}} \right| \leq C_1 e^{-d\delta} + C_2 \frac{\Gamma(N+1/2)}{\pi (d\delta)^{N+\frac{1}{2}}} \leq \frac{C(N)}{(d\delta)^{N+\frac{1}{2}}}.$$

Hence, we can write

$$n_{0,d} = e^{-dx_0} \left( \sum_{k=3}^{N-1} \frac{a_{2k-1} \Gamma(k+1/2)}{\pi i d^{k+1/2}} + O(d^{-N-1/2}) \right).$$

Since  $n_{0,d} \in \mathbb{R}$ , so does  $-ia_{2k-1}$ . Set

$$a_k^0 = \frac{a_{2k-1} \Gamma(k+1/2)}{\pi i}.$$

Thus we get the asymptotic expansion of  $n_{0,d}$  as we stated in Theorem 1.1.

## 4 Proof of Theorem 1.2

We will adopt the notations from last section. First we define a generating function

$$F_1(z) = \sum_{d=1}^{\infty} n_{1,d} e^{dz}. \quad (4.1)$$

It is proved in [Pa99] that  $F_1$  satisfies

$$(27 + 2F'_0 - 3F''_0) F'_1 = \frac{1}{8} (F'''_0 - 3F''_0 + 2F'_0). \quad (4.2)$$

By (3.5),  $\widehat{w}'(t) > 0$ , so we have  $c_1 > 0$ ,  $c'_1 > 0$  and  $a_5 \neq 0$ . Since  $(27 + 2F'_0 - 3F''_0)(x_0) = 0$  and  $a_1 = a_3 = 0$ , by (3.3), for any  $z$  with  $|z| < \delta_0$  and  $0 \leq \arg z < \frac{3}{2}\pi$ , we have

$$(27 + 2F'_0 - 3F''_0)(x_0 + z) = \sum_{d=1}^{\infty} \left( -\frac{3(d+4)(d+2)}{4} a_{d+4} + (d+2) a_{d+2} \right) z^{d/2}$$

and the expansion of  $(F'''_0 - 3F''_0 + 2F'_0)(x_0 + z)$  is

$$\sum_{d=-1}^{\infty} \left( \frac{(d+6)(d+4)(d+2)}{8} a_{d+6} - \frac{3(d+4)(d+2)}{4} a_{d+4} + (d+2) a_{d+2} \right) z^{d/2}.$$

Therefore, we can write

$$F'_1(x_0 + z) = \sum_{d=-2}^{\infty} a'_d z^{d/2},$$

where  $a'_{-2} = -\frac{1}{48}$ . Hence,  $F'_1$  is analytic in the region  $\{\operatorname{Re} z < x_0\}$  and can be analytically continued to  $\{\operatorname{Re} z < x_0 + \delta_0, 0 \leq \operatorname{Im} z \leq 2\pi, z \neq x_0, x_0 + 2\pi i\}$  for some  $0 < \delta_0 < 1$ , moreover, we have the following expansions:

$$F'_1(x_0 + re^{i\theta}) = \sum_{d=-2}^{\infty} a'_d r^{d/2} e^{i\theta d/2}, \quad \forall 0 < r < \delta_0, 0 \leq \theta \leq \pi, \quad (4.3)$$

$$F'_1(x_0 + 2\pi i + re^{i\theta}) = \sum_{d=-2}^{\infty} a'_d r^{d/2} e^{i\theta d/2}, \quad \forall 0 < r < \delta_0, \pi \leq \theta \leq 2\pi. \quad (4.4)$$

Fix  $0 < \delta < \delta_0$ , using contour integration, we have for all  $0 < \delta_1 < \delta$ ,

$$\begin{aligned}
dn_{1,d} &= \frac{1}{2\pi i} \int_{x_0-\delta_1}^{x_0-\delta_1+2\pi i} F_1'(z) e^{-dz} dz \\
&= -\frac{\delta_1}{2\pi} \int_0^\pi F_1'(x_0 + \delta_1 e^{it}) e^{-d(x_0+\delta_1 e^{it})} e^{it} dt \\
&\quad + \frac{1}{2\pi i} \int_{\delta_1}^\delta F_1'(x_0 + t) e^{-d(x_0+t)} dt \\
&\quad + \frac{1}{2\pi} \int_0^{2\pi} F_1'(x_0 + \delta + it) e^{-d(x_0+\delta+it)} dt \\
&\quad - \frac{1}{2\pi i} \int_{\delta_1}^\delta F_1'(x_0 + 2\pi i + t) e^{-d(x_0+t)} dt \\
&\quad - \frac{\delta_1}{2\pi} \int_\pi^{2\pi} F_1'(x_0 + 2\pi i + \delta_1 e^{it}) e^{-d(x_0+\delta_1 e^{it})} e^{it} dt. \tag{4.5}
\end{aligned}$$

It is easy to see

$$\left| \frac{1}{2\pi} \int_0^{2\pi} F_1'(x_0 + \delta + it) e^{-d(x_0+\delta+it)} dt \right| \leq C_1' e^{-d(x_0+\delta)}, \tag{4.6}$$

where  $C_1' = \max_{0 \leq t \leq \pi} |F_1'(x_0 + \delta + it)| < +\infty$ . By (4.3) and (4.4), as  $\delta_1 \rightarrow 0$ , we get

$$\begin{aligned}
&\frac{1}{2\pi i} \int_{\delta_1}^\delta F_1'(x_0 + t) e^{-d(x_0+t)} dt - \frac{1}{2\pi i} \int_{\delta_1}^\delta F_1'(x_0 + 2\pi i + t) e^{-d(x_0+t)} dt \\
&= \frac{1}{2\pi i} \int_{\delta_1}^\delta (F_1'(x_0 + t) - F_1'(x_0 + 2\pi i + t)) e^{-d(x_0+t)} dt \\
&= \frac{1}{2\pi i} \int_{\delta_1}^\delta \left( \sum_{k=-2}^\infty a_k' t^{k/2} - \sum_{k=-2}^\infty a_k' t^{k/2} (-1)^k \right) e^{-d(x_0+t)} dt \\
&= \sum_{k=0}^\infty \frac{a_{2k-1}'}{\pi i} \int_{\delta_1}^\delta t^{k-1/2} e^{-d(x_0+t)} dt \rightarrow e^{-dx_0} \sum_{k=0}^\infty \frac{a_{2k-1}'}{\pi i} A(k, \delta, d), \tag{4.7} \\
&- \frac{\delta_1}{2\pi} \int_0^\pi F_1'(x_0 + \delta_1 e^{it}) e^{-d(x_0+\delta_1 e^{it})} e^{it} dt \\
&- \frac{\delta_1}{2\pi} \int_\pi^{2\pi} F_1'(x_0 + 2\pi i + \delta_1 e^{it}) e^{-d(x_0+\delta_1 e^{it})} e^{it} dt \rightarrow \\
&- \frac{1}{2\pi} \int_0^\pi a_{-2}' e^{-it} e^{-dx_0} e^{it} dt - \frac{1}{2\pi} \int_\pi^{2\pi} a_{-2}' e^{-it} e^{-dx_0} e^{it} dt = -a_{-2}' e^{-dx_0}.
\end{aligned}$$

Here we have used the dominated convergence theorem. Then, arguing in a similar way as we did in the case of  $n_{0,d}$ , we can deduce

$$dn_{1,d} = e^{-dx_0} \left( -a_{-2}' + \sum_{k=0}^{N-1} \frac{a_{2k-1}' \Gamma(k+1/2)}{\pi i d^{k+1/2}} + O(d^{-N-1/2}) \right).$$

Since  $a'_{-2} = -\frac{1}{48}$ , we have

$$n_{1,d} = e^{-dx_0} \left( \frac{1}{48d} + \sum_{k=0}^{N-1} \frac{a'_{2k-1} \Gamma(k+1/2)}{\pi i d^{k+3/2}} + O(d^{-N-3/2}) \right).$$

Since  $n_{1,d} \in \mathbb{R}$ , so does  $-ia'_{2k-1}$ . Set

$$a_k^1 = \frac{a'_{2k-1} \Gamma(k+1/2)}{\pi i}.$$

Thus we get the asymptotic expansion of  $n_{0,d}$  as we stated in Theorem 1.2.

## References

- [FI94] P. Di Francesco and C. Itzykson: Quantum intersection rings, hep-th/9412175.
- [FO99] K. Fukaya and K. Ono: Arnold conjecture and Gromov-Witten invariant. Topology 38 (1999), no. 5, 933-1048.
- [LT96] J. Li and G. Tian: Virtual moduli cycles and Gromov-Witten invariants of general symplectic manifolds, Topics in symplectic 4-manifolds (Irvine, CA, 1996), 47-83, Int. Press Lect. Ser., I, Int. Press, Cambridge, MA.
- [MS94] D. McDuff and D. Salamon: J-holomorphic curves and quantum cohomology, Amer. Math. Soc. Lecture Series 6, 1994.
- [Pa99] R. Pandharipande: A geometric construction of Getzler's elliptic relation, Math. Ann. 313 (1999), no. 4, 715-729.
- [RT94] Y. Ruan and G. Tian: A mathematical theory of quantum cohomology, J. Diff. Geom. 42 (1995), no. 2, 259-367. Announcement in Math. Res. Lett. 1 (1994), no. 2, 269-278.
- [RT97] Y. Ruan and G. Tian: Higher genus symplectic invariants and sigma models coupled with gravity, Invent. Math. 130 (1997), no. 3, 455-516.
- [Zi11] A. Zinger: On Asymptotic Behavior of GW-Invariants, preprint (2011).